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EXPANSION AND ISOPERIMETRIC CONSTANTS FOR PRODUCT GRAPHS

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Vertex and edge isoperimetric constants of graphs are studied. Using a functional-analytic approach, the growth properties, under products, of these constants is analyzed.

1. Introduction

Let $G=(V,\mathcal{E})$ be a finite, undirected, connected graph. In order to study expansion properties of graphs, Alon [1] studied the so-called *expansion of* G, i.e. the isoperimetric constant h_{out}, given by

$$h_{\mathrm{out}} = \min_{A\subset V}\{|\,\partial_{\mathrm{out}}A|\,/\,|\,A|:0<|\,A|\le|\,V|\,/2\},$$

where for $A\subset V$, $\partial_{\mathrm{out}}A=\{x\notin A:\exists\, y\in A, x\sim y\}$ is the *outer vertex boundary* of A while $|\cdot|$ denotes cardinality. h_{out} is thus the minimum value of the average out-degree of a subset of G of size at most half of $|V|$. A large value of the isoperimetric number tells one that a graph does expand, at least for large (constant fraction of $|G|$) subsets – for smaller subsets the bound given by the isoperimetric number is usually far from sharp.

Recall that a k-regular graph with n vertices is called an (n,k,c)-expander, if $h_{\mathrm{out}}>c$ for some absolute constant $c>0$. In applications, one considers an infinite family of expanders with k and c fixed and n going to infinity. Usually, one prefers k to be as small as possible and it is always

Mathematics Subject Classification (2000): 05C50, 05C99, 68E10, 68Q01; 60C05, 60E15 desirable for c to be as large as possible. The explicit construction of such a family is a non-trivial task and was first achieved by Margulis [12]. In the present work we are interested in a different but related question. We consider general graphs and take graph products rather than families of graphs. As a result, the degree grows without bound and $h_{\rm out}$ could be arbitrarily small. We show that $h_{\rm out}$ of the Cartesian product of n graphs is at least $C/\sqrt{n}$, where n is the number of factors and C depends on the graph, but not on n.

Let $\partial_{\rm in} A = \{x \in A : \exists\, y \notin A, x \sim y\}$ be the *inner vertex boundary* and let $\partial_{\rm v} A = \partial_{\rm in} A \cup \partial_{\rm out} A$ be the *symmetric vertex boundary*. Note that $\partial_{\rm in} A = \partial_{\rm out} A^c$ and $\partial_{\rm v} A = \partial_{\rm v} A^c$, where $A^c = V \setminus A$ is the complement of A in V. Define the corresponding isoperimetric constants $h_{\rm in}$ and $h_{\rm v}$ [4] by

$$h_{\rm in} = \min_{A \subset V} \{|\,\partial_{\rm in} A| \,/\, |\,A| : 0 < |\,A| \le |\,V|\,/2\},$$

and

$$h_{\rm v} = \min_{A \subset V} \{|\,\partial_{\rm v} A| \,/\, |\,A| : 0 < |\,A| \le |\,V|\,/2\}.$$

Given two graphs $G_1 = (V_1, \mathcal{E}_1)$ and $G_2 = (V_2, \mathcal{E}_2)$, the Cartesian product $G_1 \square G_2 = (V, \mathcal{E})$ is defined as follows: $V = V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 while

$$\mathcal{E} = \{\{\{v_1, v_2\}, \{v_1', v_2'\}\} : v_1 = v_1' \text{ and } \{v_2, v_2'\} \in \mathcal{E}_2 \\ \text{or } \{v_1, v_1'\} \in \mathcal{E}_1 \text{ and } v_2 = v_2'\}.$$

As a corollary to our main result, we first obtain the following tensorization property of $h_{\rm in}$ whose proof simplifies and extends some recent work [10].

Theorem 1.1. *Let $G_1, \dots, G_n$ be connected graphs with Cartesian product $G^n = G_1 \square G_2 \square \cdots \square G_n$. Then*

$$h_{\rm in}(G^n) \ge \frac{\sqrt{2} - 1}{4\sqrt{n}} \min_{1 \le i \le n} h_{\rm in}(G_i). \tag{1.1}$$

Statements similar to (1.1) with $h_{\rm in}$ replaced by $h_{\rm out}$ or $h_{\rm v}$ are natural but in general, and surprisingly, false. Indeed the following two statements hold:

Theorem 1.2. *Let $G_1, \dots, G_n$ be connected graphs with Cartesian product $G^n = G_1 \square G_2 \cdots \square G_n$. Then*

$$h_{\rm out}(G^n) \ge \frac{1}{33\sqrt{n}} \min_{1 \le i \le n} \min(h_{\rm in}^2(G_i), h_{\rm out}^2(G_i)). \tag{1.2}$$

Theorem 1.3. *Let $G_1,\dots,G_n$ be connected graphs with Cartesian product $G^n = G_1 \square G_2 \cdots \square G_n$. Then*

$$h_{\mathrm{v}}(G^n) \geq \frac{1}{48\sqrt{n}} \min_{1\leq i\leq n} \min(h_{\mathrm{v}}^2(G_i), \sqrt{h_{\mathrm{v}}(G_i)}). \tag{1.3}$$

We note that all three theorems above give the correct tensorization in $1/\sqrt{n}$. The other factors, such as, $\sqrt{h_{\mathrm{v}}}$ can sometimes also represent the right order. We mention finally that solutions to the (inner, outer, symmetric) vertex isoperimetric problem for the Cartesian product of a given graph G will give precise values for $h_{\mathrm{in}}(G^n)$, $h_{\mathrm{out}}(G^n)$, and $h_{\mathrm{v}}(G^n)$. However, these isoperimetric problems are only solved in very few cases and, to the best of our knowledge, the vertex-isoperimetric problem is still unsolved even for the weighted discrete cube $\{0,1\}^n$. For the important case of down-sets, it has been solved by Bollobás and Leader [7]. The difficulty in solving such problems come from the fact that, in general, there is no ordering on the vertices of a graph such that initial segments of this ordering minimise the vertex-boundary (for sets of given size).

A brief description of the paper is as follows. In the next section we introduce various discrete gradients which unify the notions of edge and vertex boundary. In Section 3, using these gradients we prove our first "tensorization" result which in particular implies Theorem 1.1. The cases of outer and symmetric boundaries are treated in Section 4 leading to Theorem 1.2 and Theorem 1.3. We conclude by briefly discussing various generalizations and extensions of our results.

2. The discrete gradient

Let $G=(V,\mathcal{E})$ be a finite connected graph and let K be its adjacency matrix, that is, for $x,y\in V$

$$K(x,y) = \begin{cases} 1, \text{ if } x \text{ is adjacent to } y\ (x\sim y), \\ 0, \text{ otherwise.} \end{cases} \tag{2.1}$$

Let also π be a probability measure on V such that $\pi(x)>0$, $\forall x\in V$, e.g. π can be thought of as the uniform measure.

Following [10] and [4], let us consider a family of discrete gradients $\nabla_p^+ f$, $\nabla_p^- f$, $|\nabla_p f|$, for $1\leq p\leq +\infty$, (whose length is) defined for any function

$f: V \to \mathbf{R}$ and for each $x \in V$ by

$$\nabla_p^{\pm} f(x) = \left(\sum_{y \in V} \left((f(x) - f(y))^{\pm} \right)^p K(x,y) \right)^{1/p}, \tag{2.2}$$

$$|\nabla_p f|(x) = \left(\sum_{y \in V} |f(x) - f(y)|^p K(x,y) \right)^{1/p}, \tag{2.3}$$

for $p < +\infty$; and with the usual modification for $p = +\infty$:

$$\nabla_\infty^{\pm} f(x) = \sup_{y: K(x,y)>0} (f(x) - f(y))^{\pm}, \tag{2.4}$$

$$|\nabla_\infty f|(x) = \sup_{y: K(x,y)>0} |f(x) - f(y)|. \tag{2.5}$$

Let $\mathbf{1}_A$ denote the indicator function of a subset A of V. Let $\partial^+ A = \partial_{\text{in}} A = \{x \in A : \exists y \notin A, x \sim y\}$ be the *inner vertex boundary*, let $\partial^- A = \partial_{\text{out}} A = \{x \notin A : \exists y \in A, x \sim y\}$ be the *outer vertex boundary*, and let $\partial A = \partial_{\text{v}} A = \partial^+ A \cup \partial^- A$ be the *symmetric vertex boundary*. Note that $\partial^+ A = \partial^-(A^c)$, $\partial^- A = \partial^+(A^c)$, and $\partial A = \partial(A^c)$.

The following proposition elucidates the relationship between the gradients and boundaries that we have just defined.

Proposition 2.1. *(i) Let $1 \le p < \infty$ and $A \subset V$. Then*

$$\nabla_p^+ \mathbf{1}_A(x) = \mathbf{1}_{\partial^+ A}(x) \nabla_p^+ \mathbf{1}_A(x) = \begin{cases} \left(\sum_{y \in A^c} K(x,y) \right)^{1/p}, & x \in \partial^+ A, \\ 0 & , \text{ otherwise;} \end{cases}$$

$$\nabla_p^- \mathbf{1}_A(x) = \mathbf{1}_{\partial^- A}(x) \nabla_p^- \mathbf{1}_A(x) = \begin{cases} \left(\sum_{y \in A} K(x,y) \right)^{1/p}, & x \in \partial^- A, \\ 0 & , \text{ otherwise;} \end{cases}$$

$$|\nabla_p \mathbf{1}_A(x)| = \mathbf{1}_{\partial A}(x) |\nabla_p \mathbf{1}_A(x)| = \nabla_p^+ \mathbf{1}_A(x) + \nabla_p^- \mathbf{1}_A(x), x \in X.$$

(ii)

$$\nabla_\infty^+ \mathbf{1}_A = \mathbf{1}_{\partial^+ A}, \ \nabla_\infty^- \mathbf{1}_A = \mathbf{1}_{\partial^- A}, \ |\nabla_\infty \mathbf{1}_A| = \mathbf{1}_{\partial A},$$

and, therefore, $|\nabla_\infty \mathbf{1}_A| = \nabla_\infty^+ \mathbf{1}_A + \nabla_\infty^- \mathbf{1}_A$.

(iii) Moreover,

$$\nabla_p^+ \mathbf{1}_A = \nabla_p^- \mathbf{1}_{A^c}, |\nabla_p \mathbf{1}_A| = |\nabla_p \mathbf{1}_{A^c}|,$$

for any $1 \le p \le \infty$.

We see from the above proposition that $\nabla_1^+ \mathbf{1}_A(x)$ simply counts the number of edges connecting $x \in A$ with vertices in A^c. Therefore $\mathbf{E}_\pi \nabla_1^+ \mathbf{1}_A$, where $\mathbf{E}_\pi$ is expectation with respect to π, is just, when the measure is uniform, the size of the usual edge boundary of A. Analogously, $\nabla_\infty^+ \mathbf{1}_A$ is the indicator function of $\partial^+ A$, and so $\mathbf{E}_\pi \nabla_\infty^+ \mathbf{1}_A$ is the size of the inner vertex boundary of A, while $\nabla_\infty^- \mathbf{1}_A$ is the indicator function of $\partial^- A$, and $\mathbf{E}_\pi \nabla_\infty^- \mathbf{1}_A$ is the size of the outer vertex boundary of A.

A key property of the gradients in cases $p=1$ and $p=\infty$ is the so-called co-area formula (see [10], [4], [9], [14] and the references therein):

Lemma 2.2. *For any function $f : X \longrightarrow \mathbf{R}$,*

(i)

$$\mathrm{D} f = \int_{-\infty}^{+\infty} \mathrm{D}\mathbf{1}_{\{f>t\}} dt, \tag{2.6}$$

where D *denotes either one of* $\nabla_1^\pm$, $\nabla_\infty^\pm$ *or* $|\nabla_1|$;

(ii) $|\nabla_\infty f| \le \int_{-\infty}^{+\infty} |\nabla_\infty \mathbf{1}_{\{f>t\}}| dt \le 2 |\nabla_\infty f|$.

Now, let $G_i = (V_i, \mathcal{E}_i)$ be undirected graphs equipped with probability measures π_i, and let their adjacency matrices be denoted by K_i. *The Cartesian product* (G^n, π^n) of the graphs (G_i, π_i), $i=1,\dots,n$, is defined by setting $V^n = \prod_{i=1}^n V_i$, and for $x, y \in V^n$,

$$K^n(x,y) = \sum_{i=1}^n \delta(x_1,y_1)\cdots\delta(x_{i-1},y_{i-1}) K_i(x_i,y_i)\delta(x_{i+1},y_{i+1})\cdots\delta(x_n,y_n),$$

where, as usual, $\delta(x_1,y_1) = 1$ if $x_i = y_i$ and 0 otherwise; and finally, $\pi^n = \pi_1 \otimes \cdots \otimes \pi_n$.

Note that $x = (x_1,\dots,x_n) \in V^n$ is a neighbor of $y = (y_1,\dots,y_n) \in V^n$, $x \neq y$, if and only if x and y differ in one coordinate and the coordinates are neighbors in the factor graph.

For $x = (x_1, x_2, \dots, x_n) \in V^n$, we write $x^{(n-1)}$ to denote $(x_1, x_2, \dots, x_{n-1})$. Further, for $1 \le p < \infty$, let $\nabla_{p,x_n}^\pm f$, be the "positive" or "negative" one-dimensional gradient with respect to the coordinate x_n, and let $\nabla_{p,x^{(n-1)}}^\pm f$ be the "positive" or "negative" $(n-1)$-dimensional gradient with respect to the remaining coordinates $x^{(n-1)}$. They are respectively defined as:

$$\nabla^{\pm}_{p,x_n} f(x) = \Big(\sum_{y_n \in X_n} \Big((f(x) - f(x^{(n-1)}, y_n))^{\pm} \Big)^p K_n(x_n, y_n) \Big)^{1/p},$$

$$\nabla^{\pm}_{p,x^{(n-1)}} f(x) = \Big(\sum_{y^{(n-1)} \in X^{n-1}} \Big((f(x) - f(y^{(n-1)}, x_n))^{\pm} \Big)^p K^{n-1}\big(x^{(n-1)}, y^{(n-1)}\big) \Big)^{1/p},$$

and similarly for $|\nabla_{p,x_n} f|$ and $|\nabla_{p,x^{(n-1)}} f|$. In addition, define $\nabla^{\pm}_{\infty,x_n} f$ and $\nabla^{\pm}_{\infty,x^{(n-1)}} f$ with the usual modification, as well as $|\nabla_{\infty,x_n} f|$ and $|\nabla_{\infty,x^{(n-1)}} f|$.

The following properties follows directly from the definition of the Cartesian product.

Proposition 2.3. *Let* D_p *denotes either one of* $\nabla^{\pm}_p$ *or* $|\nabla_p|$, $1 \le p \le \infty$.

(i) For any $1 \le p < +\infty$ *and for* $x \in V^n$,

$$(\mathrm{D}_p f(x))^p = (\mathrm{D}_{p,x^{(n-1)}} f(x))^p + (\mathrm{D}_{p,x_n} f(x))^p,$$

and so

$$(\mathrm{D}_p f(x))^p = \sum_{i=1}^{n} (\mathrm{D}_{p,x_i} f(x))^p.$$

Moreover,

$$\mathrm{D}_\infty f(x) = \max_{1 \le i \le n} \mathrm{D}_{\infty,x_i} f(x).$$

(ii) For any $1 \le p \le +\infty$ *and* $1 \le q < +\infty$, *for* $x \in V^n$,

$$\max(1, n^{1-q/p})(\mathrm{D}_p f(x))^q \ge \max(1, (n-1)^{1-q/p})(\mathrm{D}_{p,x^{(n-1)}} f(x))^q + (\mathrm{D}_{p,x_n} f(x))^q,$$

and so

$$\max(1, n^{1-q/p})(\mathrm{D}_p f(x))^q \ge \sum_{i=1}^{n} (\mathrm{D}_{p,x_i} f(x))^q.$$

Let us now recall the definition of some isoperimetric and related constants that are associated with the gradients defined above. For any p, $1 \le p \le \infty$, let:

$$i^{\pm}_p = \inf_{0 < \pi(A) \le 1/2} \frac{\mathbf{E}_\pi \nabla^{\pm}_p \mathbf{1}_A}{\pi(A)}, \quad i_p = \inf_{0 < \pi(A) \le 1/2} \frac{\mathbf{E}_\pi |\nabla_p \mathbf{1}_A|}{\pi(A)}.$$

Note that when π is the uniform measure on V, $i^+_\infty = h_{\text{in}}$ and $i^-_\infty = h_{\text{out}}$, while $i^+_1 = i^-_1 = i(G)$ is the edge isoperimetric number of an undirected graph. In a similar way, for any p, $1 \le p \le \infty$, let

$$\tilde{i}^{\pm}_p = \inf_{0<\pi(A)<1} \frac{\mathbf{E}_\pi \nabla^{\pm}_p \mathbf{1}_A}{\pi(A)\pi(A^c)}, \quad \tilde{i}_p = \inf_{0<\pi(A)<1} \frac{\mathbf{E}_\pi |\nabla_p \mathbf{1}_A|}{\pi(A)\pi(A^c)}.$$

Below we summarize some basic and elementary properties of the constants we have just defined (see [14]).

Proposition 2.4. *(i) All the constants are positive;*
(ii) All the constants are non-increasing in p;
(iii) $\tilde{i}^+_p = \tilde{i}^-_p$*;*
(iv) For any p, $1 \le p \le \infty$,

$$\min(i^+_p, i^-_p) < \tilde{i}^{\pm}_p \le 2\min(i^+_p, i^-_p) \le i^+_p + i^-_p \le i_p < \tilde{i}_p \le 2i_p.$$

Example 2.5. (Weighted two point space) Let $\{0,1\}$ be endowed with the probability measure $\pi = q\delta_0 + r\delta_1, q + r = 1$, on its vertices. It is easy to see that for $1 \le p \le \infty$,

$$i^+_p = 1, \; i^-_p = \max\left(\frac{q}{r}, \frac{r}{q}\right), \; \tilde{i}^{\pm}_p = \min\left(\frac{1}{r}, \frac{1}{q}\right), \; i_p = \max\left(\frac{1}{r}, \frac{1}{q}\right), \; \tilde{i}_p = \frac{1}{qr},$$

where p does not play any role since the graph has only two vertices. Note that, i^-_p, i_p and $\tilde{i}_p$ can grow without bound, in contrast to i^+_p and $\tilde{i}^{\pm}_p$.

Finally, to state our main theorem in its full generality, we will need the functional versions of the isoperimetric constants. For any p, such that $1 \le p \le \infty$, define the *isoperimetric constants* j^+_p*,* j^-_p*, and* j_p by

$$j^{\pm}_p = \inf_{f \ne \text{const}} \frac{\mathbf{E}_\pi \nabla^{\pm}_p f}{\mathbf{E}_\pi (f - m(f))^+}, \quad j_p = \inf_{f \ne \text{const}} \frac{\mathbf{E}_\pi |\nabla_p f|}{\mathbf{E}_\pi |f - m(f)|},$$

where the infimum is over all non-constant $f : V \to \mathbf{R}$ and $m(f)$ is a median of f (with respect to π). Clearly, $j^{\pm}_p \le i^{\pm}_p$ and $j_p \le i_p$.

Note that for the functional equivalent of $\min(i^+_p, i^-_p)$, we have

$$\min(j^+_p, j^-_p) = \inf_{f \ne \text{const}} \frac{\mathbf{E}_\pi \nabla^{\pm}_p f}{\mathbf{E}_\pi |f - m(f)|}.$$

With the help of Lemma 2.2, the following important relations hold true (see [10], [4], [9], [14] and the references therein):

Proposition 2.6. *(i)* $i^{\pm}_1 = j^{\pm}_1$ *and* $i^{\pm}_\infty = j^{\pm}_\infty$*;*
(ii) $i_1 = j_1$ *and* $j_\infty \le i_\infty \le 2j_\infty$*.*

3. Isoperimetric Constants and Products

In this section we prove:

Theorem 3.1. *For any* $1\le p\le\infty$

$$i_p^+(G^n)\ge\frac{\sqrt{2}-1}{4}\frac{1}{\max(1,n^{1/2-1/p})}\min_{1\le i\le n}j_p^+(G_i). \tag{3.1}$$

Therefore,

$$i_\infty^+(G^n)\ge\frac{\sqrt{2}-1}{4\sqrt{n}}\min_{1\le i\le n}i_\infty^+(G_i). \tag{3.2}$$

The strategy of proof will be to study first a one-dimensional inequality and then use an induction argument. Given $I_{\mathrm{Var}}(t)=t(1-t)$, $0\le t\le 1$, and given a connected graph $G=(V,\mathcal{E})$ with a probability measure π on the vertices, for $1\le p\le\infty$, let v_p^+ be the optimal constant in the inequality

$$I_{\mathrm{Var}}(\mathbf{E}_\pi f)\le\mathbf{E}_\pi\sqrt{I_{\mathrm{Var}}(f)^2+(\nabla_p^+f)^2/{v_p^+}^2}, \tag{3.3}$$

for every function $f:V\to[0,1]$ and analogously for v_p^- and v_p. Note that $v_p^+=v_p^-$ by using $1-f$ in (3.3). Also, for indicator functions, we get the definition of $\tilde{i}_p^+$, so $\tilde{i}_p^+\ge v_p^+$ and analogously, $\tilde{i}_p\ge v_p$.

This inequality was originally proved in [2] on the two-point space with uniform measure and for the Gaussian isoperimetric function $I(t)=\varphi(\Phi^{-1}(t))$, $t\in[0,1]$, where $\varphi(t)=\exp(-t^2/2)/\sqrt{2\pi}$, $t\in\mathbf{R}$ and Φ^{-1} is the inverse of the standard normal distribution function. Nowadays, various extensions are known (see for example, [3]).

Example 3.2 (Weighted two point space continued). For $1\le p\le\infty$,

$$v_p=\sqrt{\frac{2}{qr}},\ v_p^+=v_p^-=\min\left(\sqrt{\frac{2-q}{q}},\sqrt{\frac{2-r}{r}}\right).$$

Let us show the statement for v_p (see [3] for v_p^+).

To prove that $v_p\ge\sqrt{2/(qr)}$, take a function $f:\{0,1\}\to[0,1]$ and note that $I_{\mathrm{Var}}(\mathbf{E}_\pi f)=\mathbf{E}_\pi f(1-\mathbf{E}_\pi f)\le 1/4$. Since $\mathbf{E}_\pi I_{\mathrm{Var}}(f)\le I_{\mathrm{Var}}(\mathbf{E}_\pi f)$ by Jensen's inequality, we have that

$$I_{\mathrm{Var}}(\mathbf{E}_\pi f)^2-(\mathbf{E}_\pi I_{\mathrm{Var}}(f))^2\le(I_{\mathrm{Var}}(\mathbf{E}_\pi f)-\mathbf{E}_\pi I_{\mathrm{Var}}(f))2I_{\mathrm{Var}}(\mathbf{E}_\pi f)\le\frac{\mathrm{Var}(f)}{2}.$$

Therefore

$$(I_{\mathrm{Var}}(\mathbf{E}_\pi f))^2 \leq (\mathbf{E}_\pi I_{\mathrm{Var}}(f))^2 + \frac{\mathrm{Var}(f)}{2} \leq \left(\mathbf{E}_\pi \sqrt{I_{\mathrm{Var}}(f)^2 + \mathrm{Var}(f)/2}\right)^2,$$

with the help of the triangle inequality $\mathbf{E}_\pi\sqrt{u^2+w^2} \geq \sqrt{(\mathbf{E}_\pi u)^2+(\mathbf{E}_\pi w)^2}$, applied to $u = I_{\mathrm{Var}}(f)$ and $w = \sqrt{\mathrm{Var}(f)/2}$. Now, if $f(0)=b$ and $f(1)=a$, then $\mathrm{Var}(f)=qr(b-a)^2$ and $|\nabla_p f|^2(0)=|\nabla_p f|^2(1)=(b-a)^2$. Taking square roots in the above inequality, we end up with

$$I_{\mathrm{Var}}(\mathbf{E}_\pi f) \leq \mathbf{E}_\pi \sqrt{I_{\mathrm{Var}}(f)^2 + |\nabla_p f|^2/(2/qr)},$$

which shows that $v_p \geq \sqrt{2/(qr)}$. To show that $v_p \leq \sqrt{2/(qr)}$, we use a result which will be proved later. Indeed, the first inequality in (4.7) gives $\lambda_p \geq v_p^2/2$, and in addition $\lambda_p = 1/qr$ (see Example 4.3).

Proposition 3.3. *For any undirected, connected graph,*

$$v_p^+ \geq \frac{\sqrt{2}-1}{2} j_p^+.$$

Proof. Let $N(x)=\sqrt{1+x^2}-1$. By a generalization of Cheeger's inequality (see [5], [10]) we have, for a function $f:V\to[0,1]$ with $m(f)=0$,

$$\mathbf{E}_\pi N(f) \leq \mathbf{E}_\pi N\left(\frac{2}{j_p^+}\nabla_p^+ f\right). \tag{3.4}$$

Now, the function $\psi(a)=\sqrt{a^2+x^2}-a$, $a>0$, is non-increasing in a. Therefore, from (3.4), for any $f:X\to[0,1]$ with $m(f)=0$,

$$\begin{aligned}\mathbf{E}_\pi\left(\sqrt{1+f^2}-1\right) &\leq \mathbf{E}_\pi\left(\sqrt{1+(\nabla_p^+ f)^2/c_1^2}-1\right)\\ &\leq \mathbf{E}_\pi\left(\sqrt{c^2(f(1-f))^2+(\nabla_p^+ f)^2/c_1^2}-cf(1-f)\right),\end{aligned}$$

with $c_1 = j_p^+/2$ and for some $0<c<1$ to be determined later. So

$$c\mathbf{E}_\pi f(1-f) + \mathbf{E}_\pi\left(\sqrt{1+f^2}-1\right) \leq \mathbf{E}_\pi\sqrt{c^2 I_{\mathrm{Var}}(f)^2 + (\nabla_p^+ f)^2/c_1^2}. \tag{3.5}$$

Assume that for some $c>0$ and for $f:X\to[0,1]$ with $m(f)=0$,

$$c\mathrm{Var}(f) \leq \mathbf{E}_\pi\left(\sqrt{1+f^2}-1\right). \tag{3.6}$$

Then

$$cI_{\mathrm{Var}}(\mathbf{E}_\pi f) = c\mathrm{Var}(f) + c\mathbf{E}_\pi f(1-f) \le c\mathbf{E}_\pi f(1-f) + \mathbf{E}_\pi \left(\sqrt{1+f^2}-1\right),$$

and so, by (3.5), $cI_{\mathrm{Var}}(\mathbf{E}_\pi f) \le \mathbf{E}_\pi \sqrt{c^2 I_{\mathrm{Var}}(f)^2 + (\nabla_p^+ f)^2/c_1^2}$ or

$$I_{\mathrm{Var}}(\mathbf{E}_\pi f) \le \mathbf{E}_\pi \sqrt{I_{\mathrm{Var}}(f)^2 + (\nabla_p^+ f)^2/(cc_1)^2}.$$

But inequality (3.6) is true with $c = \sqrt{2}-1$. Indeed, it is easy to check that $(\sqrt{2}-1)x^2 \le \sqrt{1+x^2}-1$, for $0 \le x \le 1$, and since $|f - m(f)| \le 1$,

$$(\sqrt{2}-1)\mathrm{Var}(f) \le (\sqrt{2}-1)\mathbf{E}_\pi (f-m(f))^2 \le \mathbf{E}_\pi \left(\sqrt{1+(f-m(f))^2}-1\right).$$

This finishes the proof of the inequality (3.3) with

$$v_p^+ \ge cc_1 = \frac{\sqrt{2}-1}{2} j_p^+. \qquad \blacksquare$$

The following lemma, inspired by a result of [2] combined with Proposition 2.3, is a key step in obtaining estimates on the isoperimetric constants of product graphs.

Lemma 3.4. *Let I be a non-negative function on $[0,1]$. Let D_p denotes either one of the gradients $\nabla_p^\pm$ or $|\nabla_p|$, $1 \le p \le \infty$. For each $i = 1, \ldots, n$, assume that the following inequality is satisfied:*

$$I(\mathbf{E}_{\pi_i} f) \le \mathbf{E}_{\pi_i} \sqrt{I(f)^2 + (\mathrm{D}_{p,x_i} f)^2/v_{p,i}^2}, \tag{3.7}$$

for every function $f : V_i \to [0,1]$, where $0 < v_{p,i}$. Then

$$I(\mathbf{E}_{\pi^n} f) \le \mathbf{E}_{\pi^n} \sqrt{I(f)^2 + c_p^{(n)} (\mathrm{D}_p f)^2/(v_p^{(n)})^2}, \tag{3.8}$$

for every function $f : V^n \to [0,1]$, where $v_p^{(n)} = \min_{1 \le i \le n} v_{p,i}$ and $c_p^{(n)} =$ $\max(1, n^{1-2/p})$.

Proof. The proof is by induction. Given $f : X^n \to [0,1]$, let $y = x^{(n-1)}$ and $f_{x_n}(y) = f(y, x_n)$ for ease of notation. Denote by $\mathbf{E}_{\pi_n}$ and $\mathbf{E}_{\pi^{n-1}}$ the expectations with respect to π_n, and $\pi^{n-1} = \pi_1 \otimes \cdots \otimes \pi_{n-1}$ respectively. From the property of D_p obtained, for $q = 2$, in Proposition 2.3 we have that

$$\begin{aligned}\mathbf{E} &\equiv \mathbf{E}_{\pi^n} \sqrt{I(f)^2 + c_p^{(n)} (\mathrm{D}_p f)^2/(v_p^{(n)})^2} \\ &\ge \mathbf{E}_{\pi_n} \mathbf{E}_{\pi^{n-1}} \sqrt{I(f)^2 + c_p^{(n-1)} (\mathrm{D}_{p,y} f)^2/(v_p^{(n-1)})^2 + (\mathrm{D}_{p,x_n} f)^2/v_{p,n}^2},\end{aligned}$$

using also $1/v_p^{(n)} \geq 1/v_p^{(n-1)}$ and $1/v_p^{(n)} \geq 1/v_{p,n}$. Next, we apply the triangle inequality

$$\mathbf{E}_{\pi^{n-1}} \sqrt{u^2+v^2} \geq \sqrt{(\mathbf{E}_{\pi^{n-1}} u)^2 + (\mathbf{E}_{\pi^{n-1}} v)^2},$$

to $u(y) = u_{x_n}(y) = \sqrt{I(f(y,x_n))^2 + c_p^{(n-1)} (\mathrm{D}_{p,y} f(y,x_n))^2 / (v_p^{(n-1)})^2}$ and $v(y) = v_{x_n}(y) = \sqrt{(\mathrm{D}_{p,x_n} f(y,x_n))^2 / v_{p,n}^2}$. We obtain

$$\begin{aligned}
\mathbf{E} &\geq \\
&\mathbf{E}_{\pi_n} \sqrt{\left(\mathbf{E}_{\pi^{n-1}} \sqrt{I(f_{x_n})^2 + c_p^{(n-1)} (\mathrm{D}_{p,y} f_{x_n})^2 / (v_p^{(n-1)})^2}\right)^2 + (\mathbf{E}_{\pi^{n-1}} \mathrm{D}_{p,x_n} f)^2 / v_{p,n}^2} \\
&\geq \mathbf{E}_{\pi_n} \sqrt{I(\mathbf{E}_{\pi^{n-1}} f_{x_n})^2 + (\mathbf{E}_{\pi^{n-1}} \mathrm{D}_{p,x_n} f)^2 / v_{p,n}^2},
\end{aligned}$$

where we used the induction hypothesis for $\mathbf{E}_{\pi^{n-1}}$ to get the second inequality. The convexity of the gradients D_p finally allows us to conclude the proof:

$$\begin{aligned}
\mathbf{E} &\geq \mathbf{E}_{\pi_n} \sqrt{I(\mathbf{E}_{\pi^{n-1}} f_{x_n})^2 + (\mathrm{D}_{p,x_n} \mathbf{E}_{\pi^{n-1}} f_{x_n})^2 / v_{p,n}^2} \\
&\geq I(\mathbf{E}_{\pi_n} \mathbf{E}_{\pi^{n-1}} f_{x_n}) = I(\mathbf{E}_{\pi^n} f),
\end{aligned}$$

from the assumption on $\mathbf{E}_{\pi_n}$. ∎

Proof of Theorem 3.1. The proof is a combination of the one-dimensional inequality from Proposition 3.3, together with the induction step of Lemma 3.4.

Let $1 \leq p \leq \infty$ and recall that, by Proposition 3.3, we have inequality (3.7) with $\mathrm{D}_p = \nabla_p^+$ and $v_{p,i} = (\sqrt{2}-1) j_p^+(G_i)/2$ for each $i = 1, \dots, n$, that is,

$$I_{\mathrm{Var}}(\mathbf{E}_{\pi_i} f) \leq \mathbf{E}_{\pi_i} \sqrt{I_{\mathrm{Var}}(f)^2 + (\nabla_p^+ f)^2 / v_{p,i}^2},$$

for every function $f : X_i \to [0,1]$, where $I_{\mathrm{Var}}(t) = t(1-t)$, $0 \leq t \leq 1$. Then, by Lemma 3.4,

$$I_{\mathrm{Var}}(\mathbf{E}_{\pi^n} f) \leq \mathbf{E}_{\pi^n} \sqrt{I_{\mathrm{Var}}(f)^2 + c_p^{(n)} (\nabla_p^+ f)^2 / (v_p^{(n)})^2},$$

for every function $f : X^n \to [0,1]$, where $v_p^{(n)} = \min_{1 \leq i \leq n} (\sqrt{2}-1) j_p^+(K_i)/2$ and $c_p^{(n)} = \max(1, n^{1-2/p})$. Hence, applying this to $f = \mathbf{1}_A$, $\phi \neq A \subsetneq V^n$, we obtain

$$\frac{(\sqrt{2}-1) \min_{1 \leq i \leq n} j_p^+(K_i)}{2 \max(1, n^{1/2-1/p})} I_{\mathrm{Var}}(\mathbf{E}_{\pi^n} \mathbf{1}_A) \leq \mathbf{E}_{\pi^n} \nabla_p^+ \mathbf{1}_A.$$

Therefore $\tilde{i}_p^+(G^n) \geq (\sqrt{2}-1) \min_{1 \leq i \leq n} j_p^+(K_i) / (2 \max(1, n^{1/2-1/p}))$ and recalling that $2 i_p^+(G^n) \geq \tilde{i}_p^+(G^n)$, finishes the proof. ∎

4. Expansion and Products

The methods of Theorem 3.1 do not seem to carry over to the cases of ∇_p^- and $|\nabla_p|$. Therefore, we need a different approach to the tensorization of i_p^- and i_p. This new approach correctly captures the dependence on the number of products n. Let $\Delta(G)$ be the maximum degree of a graph G.

Theorem 4.1. *For any* $1 \le p \le \infty$,

$$i_p^-(G^n) \ge$$

$$\frac{1}{4}\frac{1}{\max(1, n^{1/2-1/p})}\min_{1\le i\le n}\frac{\Big(\min(1-\sqrt{1-i_\infty^+(G_i)}, \sqrt{1+i_\infty^-(G_i)}-1)\Big)^2}{\sqrt{\min(i_\infty^+(G_i), i_\infty^-(G_i))+\Delta(G_i)^{2/p}}}. \tag{4.1}$$

Therefore,

$$i_\infty^-(G^n) \ge \frac{1}{4\sqrt{n}}\min_{1\le i\le n}\frac{\Big(\min(1-\sqrt{1-i_\infty^+(G_i)}, \sqrt{1+i_\infty^-(G_i)}-1)\Big)^2}{\sqrt{\min(i_\infty^+(G_i), i_\infty^-(G_i))+1}}. \tag{4.2}$$

Similarly, for the symmetric vertex boundary we have

Theorem 4.2. *For any* $1 \le p \le \infty$,

$$i_p(G^n) \ge \frac{1}{4}\frac{1}{\max(1, n^{1/2-1/p})}\min_{1\le i\le n}\frac{\Big(\sqrt{1+i_\infty(G_i)}-1\Big)^2}{\sqrt{i_\infty(G_i)+\Delta(G_i)^{2/p}}}. \tag{4.3}$$

Therefore,

$$i_\infty(G^n) \ge \frac{1}{4\sqrt{n}}\min_{1\le i\le n}\frac{\Big(\sqrt{1+i_\infty(G_i)}-1\Big)^2}{\sqrt{i_\infty(G_i)+1}}. \tag{4.4}$$

In order to prove these two theorems we recall that the Poincaré-type constants λ_p^+, λ_p^-, and λ_p are defined by

$$\lambda_p^\pm = \inf_{f\ne\text{const}}\frac{\mathbf{E}_\pi(\nabla_p^\pm f)^2}{\operatorname{Var} f}, \quad \lambda_p = \inf_{f\ne\text{const}}\frac{\mathbf{E}_\pi|\nabla_p f|^2}{\operatorname{Var} f},$$

where the infimum is over all non-constant $f: X \to \mathbf{R}$ (λ_2 is twice the smallest non-zero eigenvalue of the Laplacian $I-K$).

Example 4.3. (Weighted two point space continued) For $1 \le p \le \infty$,

$$\lambda_p = \frac{1}{qr}, \ \lambda_p^+ = \lambda_p^- = \min\left(\frac{1}{q}, \frac{1}{r}\right).$$

We need (see [4], [14]) the infinity analogs of the usual Cheeger's inequality:

$$\tilde{i}_\infty \ge \lambda_\infty \ge \frac{(\sqrt{1+i_\infty}-1)^2}{4}, \tag{4.5}$$

$$\tilde{i}_\infty^\pm \ge \lambda_\infty^\pm \ge \left[\min\left(1-\sqrt{1-i_\infty^+}, \sqrt{1+i_\infty^-}-1\right)\right]^2. \tag{4.6}$$

This last inequality (4.6) implies that

$$2\min(h_{\text{in}}, h_{\text{out}}) \ge \lambda_\infty^\pm \ge \min(h_{\text{in}}^2, h_{\text{out}}^2).$$

Finally, we relate the Poincaré-type constants to the constants v_p, introduced in (3.3).

Proposition 4.4. *For any* $1 \le p \le +\infty$

$$\lambda_p^\pm \ge \frac{v_p^{\pm 2}}{2} \ge \frac{\lambda_p^{\pm 2}}{\lambda_p^\pm + 2\Delta^{2/p}}, \ \lambda_p \ge \frac{v_p^2}{2} \ge \frac{\lambda_p^2}{\lambda_p + 2\Delta^{2/p}}, \tag{4.7}$$

where Δ denotes the maximum degree of the graph.

Proof. Let us prove the inequalities for v_p. Apply (3.3) to $f = 1/2 + \epsilon g$, $\mathbf{E}_\pi g = 0$. We have that $\mathbf{E}_\pi f = 1/2$, $I_{\text{Var}}(f) = 1/4 - \epsilon^2 g^2$ and so (3.3) becomes

$$\frac{1}{4} \le \mathbf{E}_\pi \sqrt{\left(\frac{1}{4} - \epsilon^2 g^2\right)^2 + \epsilon^2 |\nabla_p g|^2 / v_p^2},$$

which is equivalent to

$$1 \le \mathbf{E}_\pi \sqrt{1 - 8\epsilon^2 g^2 + 16\epsilon^4 g^4 + 16\epsilon^2 |\nabla_p g|^2 / v_p^2}.$$

For ϵ small, this implies

$$1 \le 1 - 4\epsilon^2 \mathbf{E}_\pi g^2 + 8\epsilon^2 \mathbf{E}_\pi |\nabla_p g|^2 / v_p^2,$$

or $\mathbf{E}_\pi g^2 \le 2\mathbf{E}_\pi (|\nabla_p g|)^2 / v_p^2$. By the very definition of λ_p, the first inequality follows.

To prove the second inequality, we present an argument inspired by [3]. Observe that for all $u \in [0, u_{\max}]$ and $w \in [0, w_{\max}]$, we have

$$\sqrt{u^2 + dw^2} - u \geq Cw^2,$$

with $d = 2Cu_{\max} + C^2 w_{\max}^2$. For $u = I_{\mathrm{Var}}(f)$ and $w = |\nabla_p f|$, note that $u_{\max} = 1/4$ and $w_{\max} = \varDelta^{1/p}$, because $0 \leq f \leq 1$. Using this and the definition of λ_p, we thus obtain

$$\begin{aligned} I_{\mathrm{Var}}(\mathbf{E}_\pi f) - \mathbf{E}_\pi I_{\mathrm{Var}}(f) = \mathrm{Var}(f) &\leq \frac{1}{\lambda_p} \mathbf{E}_\pi |\nabla_p f|^2 \\ &\leq \mathbf{E}_\pi \left(\sqrt{I_{\mathrm{Var}}(f)^2 + d |\nabla_p f|^2} - I_{\mathrm{Var}}(f) \right), \end{aligned}$$

for $d = 1/(2\lambda_p) + \varDelta^{2/p}/\lambda_p^2$. Therefore

$$v_p^2 \geq \frac{1}{d} = \frac{1}{\frac{1}{2\lambda_p} + \frac{\varDelta^{2/p}}{\lambda_p^2}} = \frac{2\lambda_p^2}{\lambda_p + 2\varDelta^{2/p}}.$$

This finishes the proof of the second inequality for v_p. The cases of $v_p^{\pm}$ are analogous and are omitted. ∎

Proof of Theorem 4.2. All we need to do is to put together several facts. Indeed, note that

$$2 i_p(G^n) \geq \tilde{i}_p(G^n) \geq v_p(G^n) \geq \frac{1}{\max(1, n^{1/2 - 1/p})} \min_{1 \leq i \leq n} v_{p,i}, \tag{4.8}$$

where the first inequality follows from Proposition 2.4, the second one is obtained using indicator functions in the version of (3.3) for the "full gradient" and the third one follows from Lemma 3.4.

Now for each of the factors, Proposition 4.4 and Cheeger's inequality (4.5) yield

$$\begin{aligned} v_p &\geq \sqrt{\frac{2\lambda_p^2}{\lambda_p + 2\varDelta^{2/p}}} \\ &\geq \sqrt{\frac{2\lambda_\infty^2}{\lambda_\infty + 2\varDelta^{2/p}}} \\ &\geq \lambda_\infty \sqrt{\frac{2}{2 i_\infty + 2\varDelta^{2/p}}} \\ &\geq \frac{\left(\sqrt{1 + i_\infty} - 1\right)^2}{4\sqrt{i_\infty + \varDelta^{2/p}}}. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) finishes the proof. Note that for $p<\infty$ better estimates can be found, but this is of no importance to us since the case $p=\infty$ is our main interest in Theorem 4.2. ∎

Theorem 4.1 can be shown in an analogous fashion and its proof is also omitted.

Proof of Theorem 1.2 and Theorem 1.3. These follow directly from Theorem 4.1 and Theorem 4.2. Indeed, (1.2) follows from (4.2), since $1-\sqrt{1-a}\geq a/2$ and $(\sqrt{1+a}-1)^2\geq(3-2\sqrt{2})\min(a^2,a)$, for $a\geq 0$, and $\min(i^+_\infty(G_i),i^-_\infty(G_i))\leq 1$. Analogously, (1.3) follows from (4.4). ∎

Example 4.5 (Weighted discrete cube). Below we show how to estimate the isoperimetic constants for $\{0,1\}^n$ directly. As before, let now G^n be the Cartesian product of n identical weighted two-point spaces $\{0,1\}$. By the very definitions of $\tilde{i}_p$, and v_p, Lemma 3.4 and Example 3.2, we have that for all $0<r<1/2$ and n,

$$\tilde{i}_p(G^n)\geq v_p(G^n)\geq\frac{v_p(G)}{\max(1,n^{1/2-1/p})}=\frac{1}{\max(1,n^{1/2-1/p})}\sqrt{\frac{2}{qr}}.$$

Hence, for all $0<r<1/2$ and n,

$$i_\infty(G^n)\geq\sqrt{\frac{2}{nqr}}. \tag{4.10}$$

Now, the Central Limit Theorem and the de Moivre–Laplace local limit Theorem show that for any $0<r<1/2$, there exists $n_0(r)$ such that for all $n\geq n_0(r)$, an upper bound for $i_\infty(G^n)$ is of order $1/\sqrt{nqr}$. This follows by arguments used in [14] to show that

$$i_2(G^n)=\Theta\left(\sqrt{\frac{1}{qr}}\right). \tag{4.11}$$

To prove the sharpness of Theorem 1.3 we now perform some more precise computation to obtain for all n and say, by symmetry, for all $0<r\leq 1/2$ such that $nr\geq 2$ is an integer, an upper bound on $i_\infty(G^n)$. To do so, recall Stirling's formula with correction (see Feller [8])

$$\sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}e^{\frac{1}{12n+1}}\leq n!\leq\sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}e^{\frac{1}{12n}}$$

Let now $X_1,\dots,X_n$ be iid Bernoulli random variables with parameter r and so $S_n=X_1+\cdots+X_n$ is a binomial random variable with parameters n

and r. If nr is an integer, then nr is the unique median of S_n (see [11]). Let $\pi^n = (q\delta_0 + r\delta_1)^{\otimes n}$, $q + r = 1$. Let us now study the isoperimetric ratio

$$\frac{\pi^n(\partial_{\mathrm{v}} A)}{\pi^n(A)}, \qquad \frac{\pi^n(\partial_{\mathrm{in}} A)}{\pi^n(A)}, \qquad \frac{\pi^n(\partial_{\mathrm{out}} A)}{\pi^n(A)},$$

for a particular set A. Take

$$A = \{x = (x_1, \ldots, x_n) \in \{0,1\}^n : x_1 + \cdots + x_n > nr\}.$$

By the very definition of the median, $\pi^n(A) = \mathbb{P}(S_n \geq nr + 1)$. Moreover, $\pi^n(\partial_{\mathrm{in}} A) = \mathbb{P}(S_n = nr + 1)$, $\pi^n(\partial_{\mathrm{out}} A) = \mathbb{P}(S_n = nr)$ and $\pi^n(\partial_{\mathrm{v}} A) = \mathbb{P}(S_n = nr) + \mathbb{P}(S_n = nr + 1)$. Let us now estimate $\mathbb{P}(S_n = nr)$ via Stirling's formula

$$\frac{1}{\sqrt{2\pi nrq}} \frac{e^{\frac{1}{12n+1}}}{e^{\frac{1}{12nr}} e^{\frac{1}{12nq}}} \leq \mathbb{P}(S_n = nr) \leq \frac{1}{\sqrt{2\pi nrq}} \frac{e^{\frac{1}{12n}}}{e^{\frac{1}{12nr+1}} e^{\frac{1}{12nq+1}}}.$$

Now, since $\mathbb{P}(S_n \geq nr + 1) \geq \frac{1}{2} - \mathbb{P}(S_n = nr)$, we get

$$\frac{\pi^n(\partial_{\mathrm{out}} A)}{\pi^n(A)} \leq \frac{1}{\sqrt{2\pi nrq}} \frac{R_1}{\frac{1}{2} - \frac{R_2}{\sqrt{2\pi nrq}}} = \frac{2R_1}{\sqrt{2\pi nrq} - 2R_2}$$

where

$$R_1 = \frac{e^{\frac{1}{12n}}}{e^{\frac{1}{12nr+1}} e^{\frac{1}{12nq+1}}}, \qquad \text{and} \qquad R_2 = \frac{e^{\frac{1}{12n+1}}}{e^{\frac{1}{12nr}} e^{\frac{1}{12nq}}}.$$

We now show that both R_1 and R_2 are bounded above by constants independent of n and p. For R_2, we have

$$4(12nr)(12nq) \leq (12n)^2 \leq (12n)(12n+1),$$

and so $R_2 \leq 1/4$. For R_1,

$$4(12nr+1)(12nq+1) \leq (12n+2)^2 \leq 24n(12n+2),$$

and so $R_1 \leq 1/2$. Combining these results, we see that for all n and r such that $nr \geq 2$ is an integer,

$$h_\infty^-(G^n) \leq \frac{2}{2\sqrt{2\pi nqr} - 1}$$

Since nr is the median, $\mathbb{P}(S_n = nr + 1) \leq \mathbb{P}(S_n = nr)$ and so

$$h_\infty(G^n) \leq \frac{4}{2\sqrt{2\pi nqr} - 1}.$$

Now, Theorem 1.3 gives us for all n and r,

$$h_\infty(G^n) \geq \frac{1}{48\sqrt{nqr}},$$

while Theorem 1.2 gives

$$h_\infty^-(G^n) \geq \frac{1}{33\sqrt{n}}.$$

Performing similar computations for $h_\infty^+(G^n) \leq C(n,r)/\sqrt{nqr}$, where now for fixed r, $C(n,r)$ is uniformly bounded in n. Theorem 1.1 gives

$$h_\infty^+(G^n) \geq \frac{\sqrt{2}-1}{4\sqrt{n}}.$$

5. Concluding Remarks

• In view of the framework developed in the references of the present paper, results similar to the ones given above continue hold, with easy modifications, in a Markov chain setting.

• (Expansion for paths and related graphs) Let P_k be the graph with vertices $V = \{0,1,\ldots,k-1\}$ where $\{i,i+1\}$ $(i = 0,\ldots,k-2)$ are the only pairs of connected vertices. It is easy to see that $h_{\text{out}}(P_k) = 2/(k-1)$ for k-odd, and that $h_{\text{out}}(P_k) = 2/k$ for k-even. In addition, $h_{\text{in}}(P_k) = h_{\text{out}}(P_k)$ and $h_{\text{v}}(P_k) = 2h_{\text{out}}(P_k)$. Bollobás and Leader [6] solved the vertex isoperimetric problem on the product of n paths P_k^n in the following sense. For any subset $A \subset V$ define *the (1-) enlargement* $A^{(1)}$ of A by $A^{(1)} = A \cup \partial_{\text{out}} A$. Now, given a subset size, find the subsets with smallest enlargements. Using these isoperimetric sets one recovers the following known and easy fact:

$$h_{\text{out}}(P_k^n) \geq \frac{1}{8}\frac{\min_{1\leq i\leq n} h_{\text{out}}(P_k)}{\sqrt{n}}.$$

An analogous result can be stated for the Cartesian product of graphs with the same number of vertices, at least one of which is "like" a path (see [14]).

• (Threshold property of the probability of disconnecting a graph by removing vertices) Margulis [13] considered the problem of disconnecting a connected graph by removing edges randomly with probability p. The probability of disconnecting a graph as a function of p experiences a "threshold behavior" (see also Talagrand [15]) – it jumps from 0 to 1 in a short interval, that shrinks as n grows. It turned out that establishing an isoperimetric inequality and using the Margulis–Russo identity is enough to prove this phenomenon. We can consider the same problem for vertices, i.e. investigate the probability of disconnecting a graph by removing vertices with probability p. It can be shown that this function demonstrates threshold behavior in an analogous fashion.

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